THE DEFORMATION OF SUB-GLACIAL TILL: A NUMERICAL MODEL

by

Robert Mellors

Senior thesis

presented in partial fulfillment of the requirements
for the degree of Bachelor of Science with distinction in
Geology and Mineralogy

The Ohio State University
Spring, 1986

Approved by:

Ian M. Whillans

Dr. Ian M. Whillans
Advisor

Acknowledgements

I wish to thank Dr. Ian Whillans for all his time, patience, and advice. He always made time for a last minute counseling session and I appreciate the marvellous opportunity to investigate the field of glaciology, in the office and in the field. I also wish to thank Richard Alley for his suggestions and the many lists of references. In addition, Dr. John Bolzan and Patricia Vornberger never hesitated to help when asked, even when busy. I also thank Dr. Scott Bair and Dr. Gerald Newsom for serving on my examining committee and for their helpful comments.

Table of Contents

Abstract

Prior work has shown that ice streams in West Antarctica are probably underlain by a layer of till 6 to 12 m thick. This till layer may be similiar to mobile drift found by Kamb (1979) on Blue Glacier in Washington State and by Boulton (1979) at Breidamerjokull in Iceland. This paper presents a preliminary model for flow of this till layer. The till is assumed to be linearly viscous with a calculated viscosity of 3X10**8 Pa s. Flow is caused by shear stress from the ice above and by pressure differences. The thickness over time is tested by solving equations numerically using a time-centered finite-difference method. Results show that the till layer tends towards a constant thickness despite initial gradients in thickness.

Introduction

One of the major topics in glaciology is the nature of ice flow in the rapidly moving ice streams of West Antarctica (figure 1). These ice streams, which are 'rivers' of ice that drain the West Antarctic ice sheet, flow towards the Ross ice shelf at velocities up to 850 m/a (Whillans,1986). It is believed that these ice streams play a crucial role in determining the stability of the West Antarctic ice sheet. Should the sheet be unstable, the resulting break-up could lead to a catastrophic rise in sea level.

Most glaciers and ice streams move by a combination of movement along the bed and by deformation of the ice itself (Paterson,1981). In fast-moving ice streams, the majority of the movement is believed to occur along the bed (Paterson,1981). Generally, it has been assumed that the glacier slides over the bedrock with a film of water in between. Several mathematical models of this flow have been developed, although none are completely satisfactory (Weertman,1979). However, recent studies by Kamb (1978) on Blue Glacier in Washington State and by Boulton (1979) on Breidamerjokull in Iceland have led to another theory. This theory suggests that the till under a moving glacier may actively deform and account for a major proportion of the glacial movement. Up to 80% of the total movement may be due to sub-glacial deformation (Boulton,1979).

This is significant because it may explain the movement of the ice streams in West Antarctica. Calculations indicate that internal deformation alone cannot account for the observed velocities and some other mechanism is necessary. Recent seismic surveys (Blankenship et al.,1986) have shown that a layer of till up to 12 m thick lies under ice stream B. This till layer is thought to be actively deforming under drag stress from the ice above (Alley et al.,1986). Very little is known about the physical properties or behavior under stress of such a till layer and consequently the effect on the ice stream by a mobile till is uncertain. An accurate model of the till would aid greatly in understanding the flow of the ice streams and also in determining the stability of the West Antarctic ice sheet.

Therefore, I have developed a model of the till layer. The major goal of the model is to identify what conditions are necessary to ensure stability over time of a deforming till layer. The till is assumed to be subject to shear stress and pressure from the overlying ice. The purpose is to determine whether the till layer will remain stable under the specified conditions of shear stress, pressure gradient, viscosity, and thickness over time.

Methods

The first step in the development of a model is to derive equations that will describe the behavior of the till layer.

These equations result from a consideration of the forces acting on the layer and from physical characteristics of the layer itself.

The forces causing the till to deform must be determined. To simplify the model, a two dimensional vertical plane parallel to the direction of flow is considered (figure 2). The horizontal coordinate is x and z is the vertical coordinate. The distance covered by the model in the x direction is 5 km, with the origin upstream. The two major forces driving the till derive from changes in pressure and shear stress from the moving ice.

The pressure (P) on the till is equal to the weight of the ice above:

$$P = rgH \tag{1a}$$

where r is the density of the ice, g is the acceleration due to gravity, and H is the thickness of the ice. Changes in the thickness of the ice over distance produces a pressure gradient dP/dx:

$$\frac{dP}{dx} = \frac{rg(H1 - H2)}{(x1 - x2)} \tag{1b}$$

in which H1-H2 is the change in ice thickness over a change in distance (x1-x2). Values for these constants are known. If the change in elevation of the till bed is known, then the average potential (p) of the till would be used which includes pressure effects as well as the till elevation:

$$p = Rg(E + h/2) + rgH. \tag{1c}$$

where R is the density of the till, E is the bed elevation above the lowest bed elevation, and h is the till thickness. This allows for the flow of till due to gravity down a slope and could replace P in equation 1b, but this term is left out in this work. In effect the till is assumed not to change elevation.

The till also flows as a response to shear stress from the ice above. The ice flows according to surface slope and that produces a shear stress T at the base of the ice. It is assumed that the flow of the ice is plug flow, or the ice moves at a constant velocity at all depths so that measured surface velocities can be used at depth. It has also been shown (Whillans 1986) that relatively little stress is absorbed at the sides or ends of the ice stream. Therefore, most of the driving stress is balanced by shear stress at the base of the glacier. Because the till layer is frozen to the base of the ice and is secure against bedrock (Boulton,1979) the shear stress must cause deformation in the till layer itself. Using the previous coordinate system and the definition of viscosity:

$$T = m \frac{du}{dz} . \qquad \text{2a}$$

The shear stress is equal to the viscosity (m) times the gradient in till velocity (u) over the layer. If this were the only stress acting on the layer, then the velocity of the till would decrease linearly from U, the velocity of the ice at the top of the till, to 0 at the base of the till, and the average velocity would be U/2. However, the pressure gradient can have an effect.

To account for the flow due to the changes in pressure, the flow is assumed to be have a zero net force balance. This is not completely correct, but the rate of change is assumed to be small. This produces no change in total momentum. Therefore, the horizontal gradient dP/dx in pressure must be balanced by a vertical gradient in shear stress T . Shear stress acting in the y or z directon is taken to be zero. Pressure varies in the z direction, but the pressure is hydrostatic, so no motion occurs. This leads to :

$$\frac{dT}{dz} = \frac{dP}{dx} \qquad \text{3a}$$

Equation 2a and 3a can be combined. If the pressure gradient is now assumed constant over the distance covered in the model and represented by I. Now combine 2a and 3a to eliminate T:

$$\frac{d(m\ du/dz)}{dz} = I \quad . \qquad \text{3b}$$

The viscosity is assumed constant and the equation is integrated, which leads to:

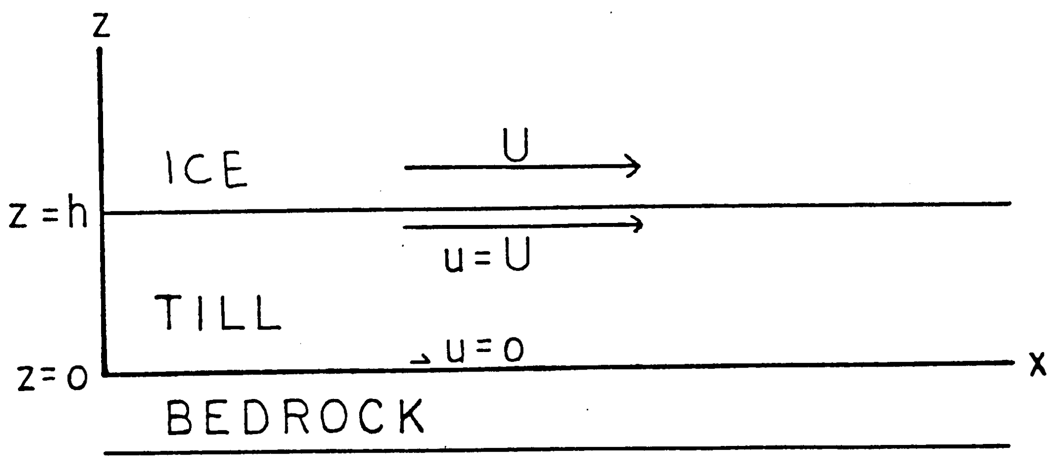
$$\frac{du}{dz} = \frac{1}{m} Iz + A \qquad \text{3c}$$

Integrate again,

3d $$u = \frac{1}{2m} Iz**2 + Az + B$$

The constants,A and B can be determined from boundary conditions.

Boundary conditions :



At z = 0 the till is held by friction against the bedrock, so u = 0. This implies implies that B = 0. Since u = U at z = h, we get:

3e $$U = \frac{1}{2m} I\ h + Ah$$

Solve for A

3f $$A = \frac{U}{h} - \frac{Ih}{2m}$$

Substitute into 3d

3g $$u(z) = \frac{1}{2m} Iz**2 - \frac{1}{2m} Ihz + \frac{Uz}{h}$$

3h $$u(z) = \frac{Uz}{h} + \frac{1}{2m} I \ (z**2 - hz)$$

If u(z) is integrated from the top of the till h to the bottom and multiply by 1/h, the result is the mean flow rate ($\bar{u}$).

4a $$\bar{u} = \frac{1}{h} \int_0^h u(z) \ dz$$

Substitute 3g for u(z)

4b $$\bar{u} = \frac{1}{h} \int_0^h \frac{Uz}{h} \ dz + \frac{1}{h} \int_0^h \frac{-1}{2m} Izh + \frac{1}{h} \int_0^h \frac{1}{2m} Iz$$

$$\bar{u} = \frac{U}{2} - \frac{h}{4m} I + \frac{1}{6m} I \ h**2$$

4c $$\bar{u} = \frac{U}{2} - \frac{h**2}{12m} I$$

This now gives the mean velocity of the till as a function of the ice velocity U, potential gradient I, till thickness h, and viscosity. All of these quantities are known or can be specified.

A value for the viscosity is obtained by differentiating equation 3g with respect to z.

5a $$\frac{du}{dz} = \frac{U}{h} + \frac{I(2z-h)}{2m}$$

Substitute from equation 2:

5b $$\frac{T}{m} = \frac{U}{h} + \frac{I(2z - h)}{2m}$$

Let z = h at the top of the till,and T = T_o and rearrange terms.

5c $$T_0 h = Um + \frac{Ih}{2}$$

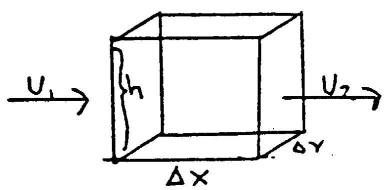
Solve for m .

5d $$m = T_0 - \frac{I}{2}\ \frac{h}{U}$$

This gives a value for the viscosity of the till.

The purpose of this model is to determine the stability over time of the till. This requires relating the thickness of the till to the velocity and for this the concept of continuity of flow is used.

Continuity describes conservation of mass and volume (if the density is invariant). It relates the thickness of the layer to the change in flow. The thickness of a layer is proportional to the ratio of the flow in to the flow out.



If the flow in (u1hy) is greater than the flow out (u2hy) then the block must increase in volume.

$$\Delta x\ y\ \frac{dh}{dt} = u2hy - u1hy$$

Divide by Δx

$$\frac{dh}{dt} = \frac{u2h - u1h}{\Delta x}$$

Take the limit as $\Delta x \to 0$

$$\frac{dh}{dt} = \frac{-d(hu)}{dx}$$

This gives the time-rate of till thickness change as a function of h and u. As the till erodes the bedrock, more mass will be added and h will increase. Therefore, erosion rate (e) is added.

$$\frac{dh}{dt} = -\frac{d(hu)}{dx} + e \qquad (6)$$

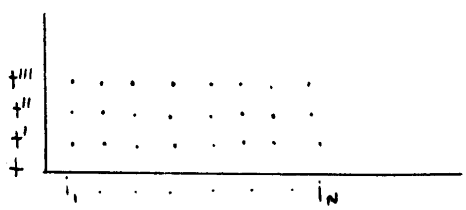
In all calculations e is set to zero.

Velocity of the till is known from 4c, so combining 6 and 4c:

$$\frac{dh}{dt} = \frac{-d}{dx}\left[\frac{hU}{2} - \frac{h{**}3\ I}{12m}\right] + e \qquad (7a)$$

Now the till height is expressed in terms of pressure gradient, present thickness, ice velocity, and erosion rate. It must be solved, either analytically or numerically. An analytic solution could not be found except for special cases so a numerical solution is used.

The Crank-Nicholson method is used.This involves solving the equation along a grid of points $i_1 \ldots\ldots i_N$ along the x axis and for time intervals of t,t',t''.



$\underline{dh}$ is expessed as $\underline{h' - h}$ where h' is the till thickness one time step ahead, so the left hand side of equation 7a can be written:

$$\frac{h' - h}{\Delta t}$$

The right hand side is approximated by averaging the change between i+1 and i-1 over 2 x. Averages are taken over a time step.

$$\frac{h'-h}{t} = \frac{1}{2}\left[\left(\frac{\left[\frac{h'U}{2}-\frac{(h')I}{12m}\right] - \left[\frac{h'U}{2}-\frac{(h')I}{12m}\right]}{2\Delta x}\right) + \left(\frac{\left[\frac{hU}{2}-\frac{(h)I}{12m}\right] - \left[\frac{hU}{2}-\frac{(h\ )I}{12m}\right]}{2\Delta x}\right)\right] + e$$

This approximates the value at i,h . Since only the values at t are known, the h' must be factored out.

$$h' - \frac{t}{8\Delta x}\left[\left(h'U - \frac{(h'\ )I}{12m}\right) - \left(h'U - \frac{(h')I}{12m}\right)\right] = \frac{t}{8\Delta x}\left[\left(hU - \frac{(h\ )I}{12m}\right) - \left(hU - \frac{(h'\ )I}{12m}\right)\right] + h + e$$

It is impossible to factor h'- k(h')**3 where k is a constant. Therefore, (h')**3 is approximated by (h)**2 h'. This results in a weighted time average closer to h than h'.

Right side: $\frac{t}{8\Delta x}\left[\left(h\ U - \frac{(h\ )I}{12m}\right) - \left(h\ U - \frac{(h\ )I}{12m}\right)\right] + h + e$

Left side: $h' - \frac{t}{8\Delta x}\left[\left(U - \frac{(h\ )I}{12m}\right) h' - \left(U - \frac{(h\ )I}{12m}\right) h'\right]$

A solution of this equation must be made at every i on the x axis except at the ends. If h is specified for every $i_1 \ldots\ldots i_N$ at time = t anf for the initial i and last i for $t' \ldots\ldots t^N$, all $h' \ldots\ldots h^N$ can be solved for. The easiest way to solve this set of equations is with a matrix.

$$\begin{bmatrix} B & C & 0 & \cdots\cdots & 0 \\ A & B & C & \cdots\cdots & 0 \\ A & B & C & \cdots\cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & 0 \\ 0 & \cdots\cdots & & A & B \end{bmatrix} \times \begin{bmatrix} h' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ h' \end{bmatrix} = \begin{bmatrix} D \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ D \end{bmatrix}$$

The h'are unknown but the A,B,C and D are known coefficients from 8 c. If the D matrix is mutiplied by the inverse of the A,B,C matrix, the h'values can be obtained. These h' are the new values for the till at the next time step.

Results

The model is tested by keeping the thickness of the till constant with an erosion rate of zero. This provides a constant flow of till at all points and no secular change in thickness occurs.

Variations on these runs with constant thickness can be used to determine the relative effects of pressure gradient and drag stress on the till. Considering traction alone, with the pressure gradient set at zero, the average velocity of the till is one half the ice velocity, as expected from the assumed linear viscous nature of the till and from the boundary conditions. If the specified presure gradient,(90 N/m), is included, the velocity of the till increases by 27% for a viscosity of $1X10^{8}$ Pa S. This shows that the pressure gradient does cause some flow although not as much as the shear stress.

Since the flow due to the pressure gradient varies inversely with the viscosity, the component of flow due to the pressure flow is insignificant (less than 3%) at viscosities greater than $1X10^{9}$ Pa S.

The effect of perturbations in the till is then tested. A sine function simulates a local 1 m thickening and 1 m thinning of the till in the middle 2 km (figure 3). The perturbation travels downstream at a rate of 360 m/a and increases in amplitude by .5 m.

The increase in amplitude is due to the relationship between the velocity and thickness of the till. The till moves faster where thick and slower when thin. This causes a build-up of the till to occur when a section of faster moving till overruns a section of slower moving till.

This increase in amplitude may not be completely realistic, as there are several feedback effects that are not included in this model. For example, a thickening of the till requires that the ice above be lifted with a corresponding sinking of the ice where the till thins. The stiffness of the ice would tend to dampen any rapid fluctuations in the thickness of the till. Another feedback effect is that the the shear stress on the till and the viscosity of the till are dependent on the till thickness. Furthermore, the elevation of the till is changed and that can lead to amplitude reduction. These processes may be diffusive and act to decrease the till thickness.

Conclusions

Modelling of the till as a linear-viscous layer driven by traction from the ice above and pressure gradients show that the traction from the ice is the dominant force in causing the till to flow. Experiments with perturbations show that instbilities develop such that the wavelength decreases as the amplitude increases over time. These results may have been associated with sinplification of the model. For a first model the results are valuable, as they show that instabilities in the till layer must be considered more carefully and that pressure gradients can be potentially important as a driving force.

References

Alley, R. B., Blankenship, D. D., Rooney, S. T., and Bentley, C. R., Ice Stream Flow on a Deformable Bed: A First Model, presented at Chapman Conference on Fast Glacier Flow, British Columbia, May 4-8, 1986.

Blankenship, D. D., Alley, R. B., Rooney, S. T., and Bentley, C. R., Seismic Measurements Reveal a Saturated, Porous Layer Beneath an Active Antarctic Ice Stream, presented at Chapman Conference on Fast Glacier Flow, British Columbia, May 4-8, 1986.

Boulton, G. S., 1979, Processes of Glacial Erosion, Journal of Glaciology, vol. 23, no. 89, p. 15-38.

Kamb, B., and Engelhardt, H. F., 1979, The Ice-Rock Interface and Basal Sliding Processes as Revealed by Direct Observation in Bore Holes and Tunnels, Journal of Glaciology, vol. 23, no. 89, p. 416-417.

Paterson, W. S. B., 1981, The Physics of Glaciers, New York, Pergamon Press, p. 112-113.

Weertman, J., 1979. The Unsolved General Glacier Sliding Problem, Journal of Glaciology, vol. 23, no. 89, p.97-115.

Whillans, I. M.,1986. Personal communication.

```
//IS5577 JOB 'SDX350,328706683','MELLORS',
// REGION=512K,TIME=(0,20)
/*JOBPARM LINES=10000
//STEP1 EXEC FORTVCG,PARM.FORT='LANGLVL(77)'
//FORT.SYSIN DD *
C          THIS PROGRAM DETERMINES THE THICKNESS OF A LAYER OF
C      DEFORMABLE TILL ACCORDING TO TIME AND OVER A CERTAIN
C      DISTANCE BY MEANS OF A CRANK-NICOLSON SOLUTION.
C
C        VARIABLE LIST
C
C        H        THICKNESS OF TILL (METERS)
C        VIS      VISCOSITY OF TILL (PASCAL YEARS)
C        ERODE    EROSION RATE (METERS)
C        PHI      CHANGE IN POTENTIAL
C        VELICE   VELOCITY OF ICE (METERS/YEARS)
C        VELTIL   VELOCITY OF TILL
C        DELX     CHANGE IN X (YEARS)
C        DELT     CHANGE IN TIME  (YEARS)
C        INXS     NUMBER OF X STEPS
C        ITIME    NUMBER OF TIME STEPS
C        A,B,C,D COEFFICIENT VALUES
C
C     ********************************************
C      DIMENSION ARRAYS
C      DOUBLE PRECISION OF VARIABLES
C
C
       DIMENSION H(100),A(100),B(100),QQ(100),C(100),D(100)
       DIMENSION BB(100),RR(100)
      DOUBLE PRECISION H,A,C,D,BB,QQ,RR,PHI,VIS,VELICE
C
C         INITIALIZE TILL THICKNESS, POTENTIAL,VISCOSITY AND ICE VELOCITY
C         AT TIME ZERO.
C
       INXS = 100
       INXS1 = INXS - 1
       INXS2 = INXS - 2
C
C            THIS INITIALIZES H OVER X
C
       DO 5 I = 1,INXS
             H(I)=6.
             IF((I.GE.40).AND.(I.LE.60))THEN
                  H(I)=H(I)+(SIN(6.28318*(I-40)/20))
             ENDIF
    5  CONTINUE
C
C   PRINT INITIAL VALUES OF H
C
        WRITE (6,6) (H(I),I=1,INXS)
 6     FORMAT('1',5(2X,D12.7))
      PHI = 920.*9.8*(-10./1000.)
      VIS = 1.0E+8/(365.25*60*60*24)
      ERODE = .000
      VELICE = 450.
      DELX = 50.0
      DELT=.01
      ITIME = 500
```

```
              VELTIL = (VELICE/2)-(H(1)**2*PHI/(12*VIS))
               PRINT*,'PHI',PHI,'VELICE',VELICE,'VELTIL',VELTIL
      C           H IS SOLVED BY MEANS OF A MATRIX MULTIPLICATION
      C           I + 1 = INXS
      C
      C                        B(2)H(2) + C(2)H(3) = D(2)
      C            A(3)H(2) + B(3)H(3) + C(3)H(4) = D(3)
      C             *                               *
      C             *                               *
      C             *                               *
      C            A(I)H(I-1)+B(I)H(I) =D(I)
      C
      C            DEFINE COEFFICIENT INSIDE TIME STEP AT DO 10
      C
      C
      C         START TIME STEP
      C
                DO 10 J = 1,ITIME
                    B(2) = 1.0
                    C(2)=(DELT/(DELX*8))*(VELICE-(H(3)**2*PHI)/(12*VIS))
                    D(2)=(-DELT/(DELX*8))*((H(3)*VELICE-H(3)**3*PHI/(12*VIS))
           @         -(H(1)*VELICE-H(1)**3*PHI/(12*VIS))) +H(2)
           @         +((DELT/(DELX*8))*(VELICE-(H(1)**2*PHI)/(12*VIS)))*H(1)
                    BB(2) = C(2)/B(2)
      C
      C
                    DO 15 J1 = 3,INXS2
                     A(J1)=(-DELT/(8*DELX))*(VELICE-(H(J1-1)**2*PHI)/(12*VIS))
                      B(J1) = 1.0
                     C(J1)=(DELT/(8*DELX))*(VELICE-(H(J1+1)**2*PHI)/(12*VIS))
                    D(J1)=(-DELT/(8*DELX))*((H(J1+1)*VELICE-H(J1+1)
           @             **3*PHI/(12*VIS))-(H(J1-1)*VELICE-
           @             H(J1-1)**3*PHI/(12*VIS)))+H(J1)
                        BB(J1)= C(J1)/(B(J1)-(A(J1)*BB(J1-1)))
       15          CONTINUE
                 B(INXS1) = 1.0
                 A(INXS1)=(-DELT/(DELX*8))*(VELICE-(H(INXS2)**2*PHI)/(12*VIS))
               D(INXS1)=(-DELT/(DELX*8))*((H(INXS)*VELICE-H(INXS)**3*PHI/(12*
           @   VIS))-(H(INXS2)*VELICE-H(INXS2)**3*PHI/(12*VIS)))+H(INXS1)
           @   -((DELT/(DELX*8))*(VELICE-H(INXS)**2*PHI/(12*VIS)))*H(INXS)
      C
      C         MUTIPLY BY MATRIX INVERSION TO DETERMINE NEW PHI
      C
                 RR(2) = D(2)/B(2)
                     DO 20 I = 3, INXS1
                           RR(I) = (D(I)-A(I)*RR(I-1))/(B(I)-A(I)*BB(I-1))
        20      CONTINUE
                           H(INXS1) = RR(INXS1)
                           DO 25 J2 = 2, INXS2
                                 I1=INXS-J2
                                 H(I1) = RR(I1) - BB(I1)*H(I1+1)
       25                  CONTINUE
      C        ADD EROSION RATE
               DO 33 J3 = 2,INXS1
                   H(J3) = H(J3) + ERODE*DELT
        33     CONTINUE
      C
      C            PRINT RESULTS
      C
               IF((J/50.) .EQ. (INT(J/50)))THEN
```

```
                  WRITE (6,50)
       50         FORMAT ('1','THICKNESS OF TILL'//)
               WRITE (6,60)(H(I),I=1,INXS)
       60      FORMAT('1',5(2X,D12.7))
               ENDIF
       10        CONTINUE
                  STOP
                  END
      /*
```

Figure 1a. Map of Antarctica with ice stream area shaded

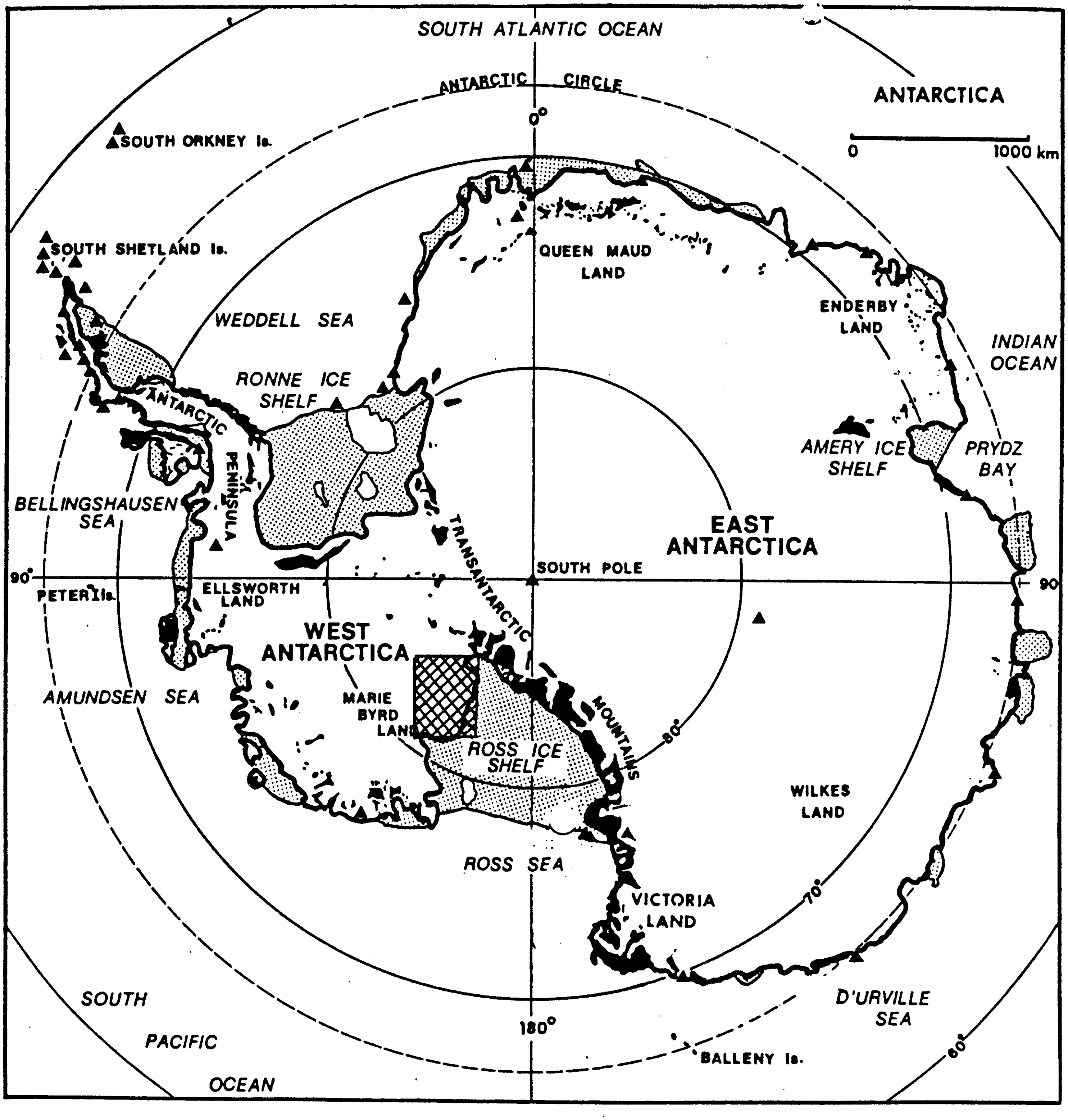
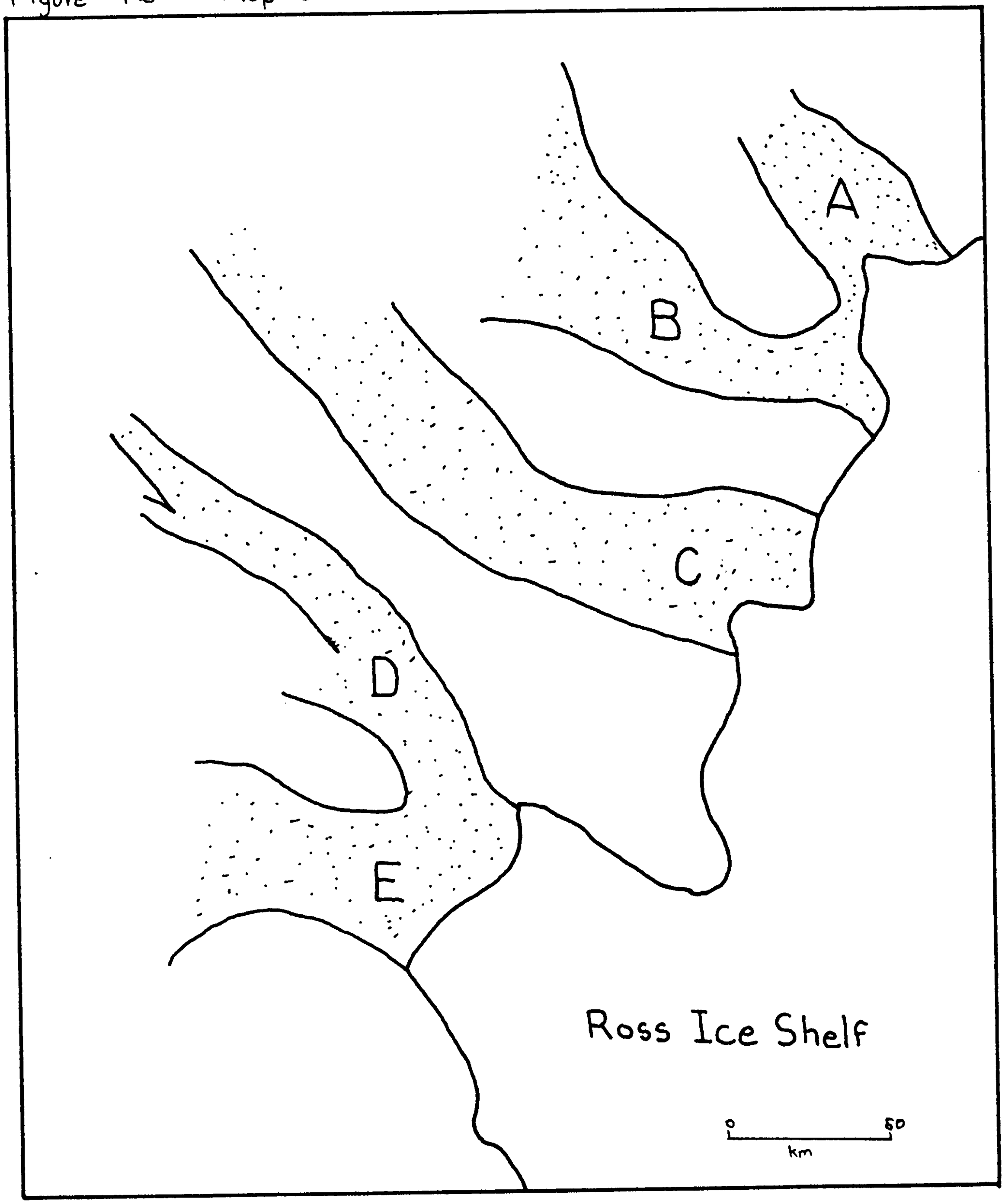


Figure 1b Map of ice streams



(adopted from Alley, et al., 1986)

Figure 2 Till model and coordinate system

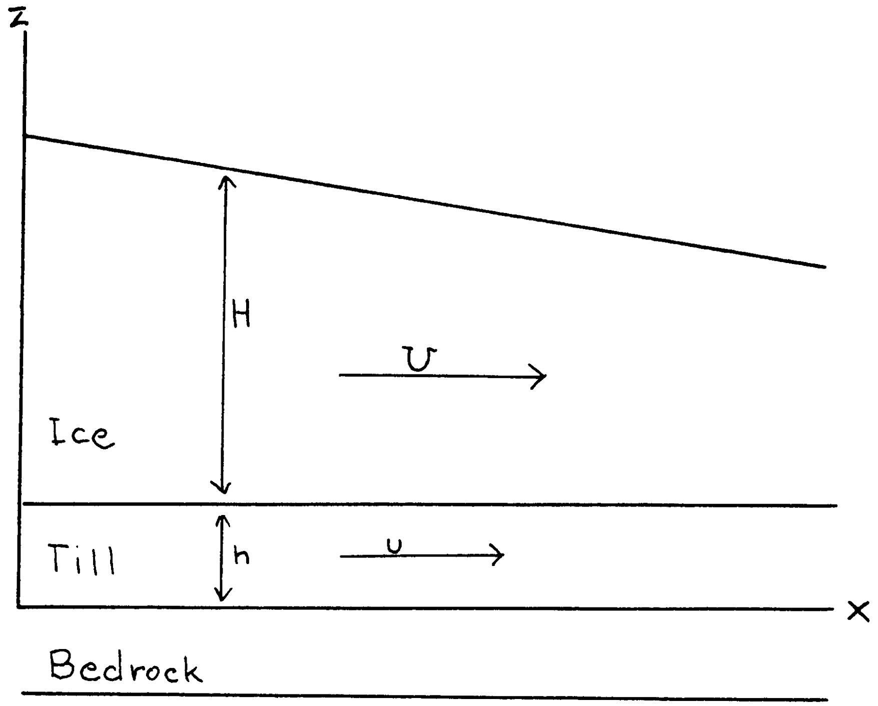
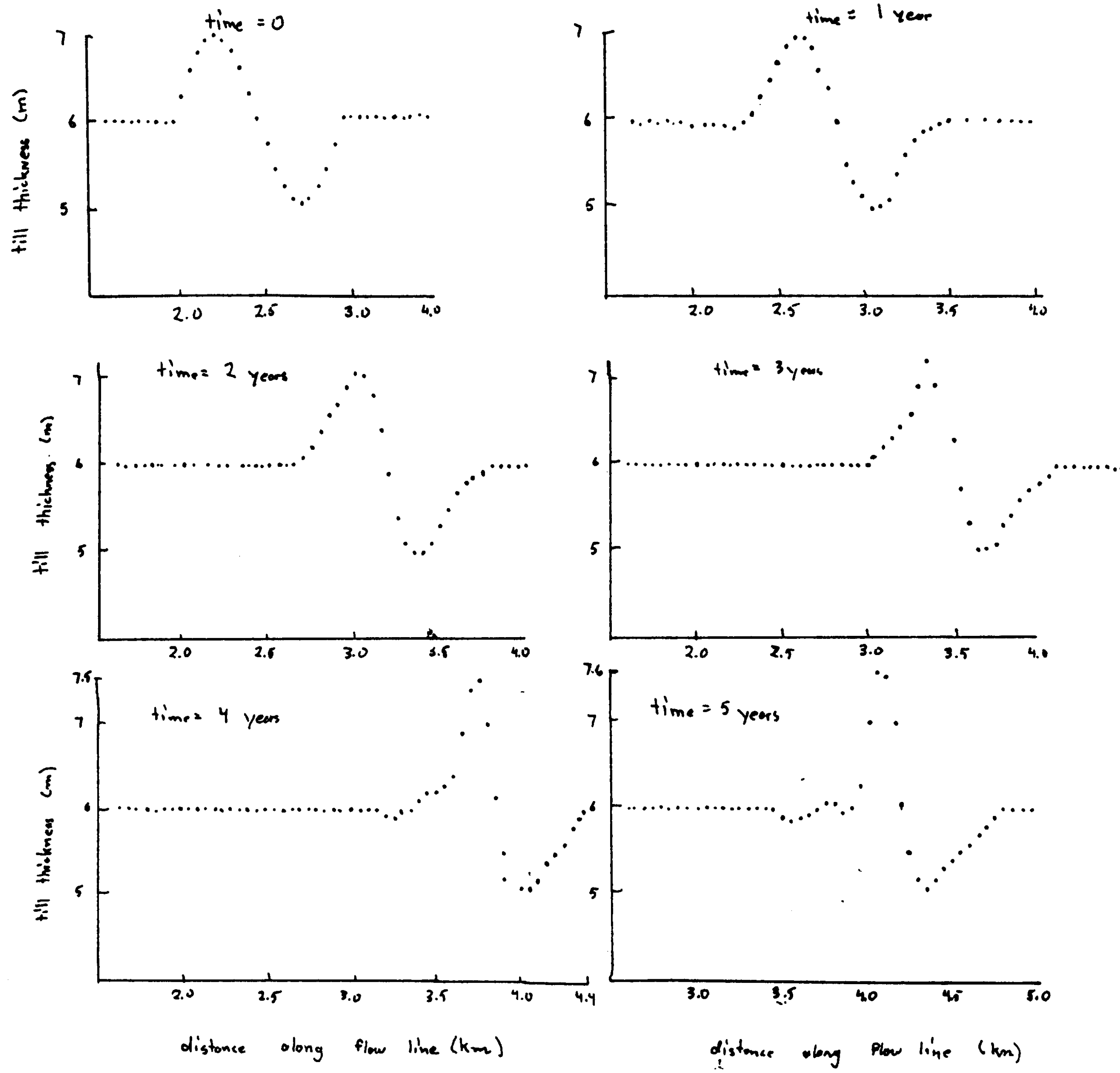


Figure 3 Results of Perturbations in till thickness



delt = .001

delx = 50 m

ice velocity = 450 m/a

till velocity (6 m) = 310 m/a

wave velocity ≈ 350 m/a